Table 1 Local element deflection, slope, and bending moments

X_{el}, m	Y_{el}, m	θ_{el}, deg	M, N-m
15.2	0.15	0.1	-10^3 to -10^4
50.3	0.15	1.0	10^4 to 10^5

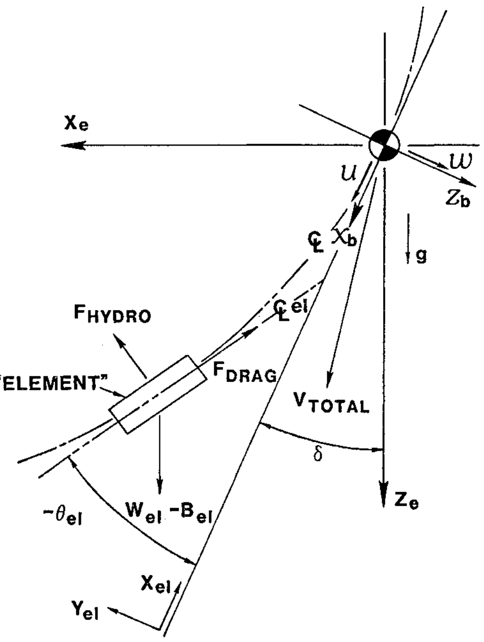

Fig. 2 Illustration of global and local coordinate systems.

m/s and an initial rotation angle of $\delta=3.0$ deg. All other dependent global variables are initially zero, as are the deflections and velocities for the local equations of motion.

Discussion of Results

The equations of motion were numerically integrated utilizing a total of 36 elements to define the geometry of the corer. The corer was allowed to free-fall for 3 s, at which time the global variables had the following values: $u=23.6$ m/s, $w=2.3$ m/s, $q=0.05$ rad/s, $\delta=6.4$ deg, $X_e=-0.8$ m, $Z_e=37.0$ m. The corer assumed an elongated inverted-S shape.

At $t=3$ s, maximum bending moments occurred at the location of the step taper and at the junction between the afterbody and the upper portion of the barrel. The local element deflection, slope, and bending moment at these locations are given in Table 1 and are approximate due to the local variables being measured at discrete locations along the length of the corer. It is unlikely that the corer will actually experience such large bending moments during free-fall. However, it is intuitive that large bending moments will occur at the barrel/afterbody junction and may even become larger during penetration.

Conclusions

Since most of the mass is concentrated at the upper end of the body, the system is inherently unstable. However, the system inertia is apparently so high that the transient response is somewhat sluggish for the drop times of interest. Non-negligible deflections and bending moments are encountered prior to penetration.

It is not yet known how instabilities and local deflections affect bending response during penetration. The dynamic behavior of the corer as it enters the sediment will be analyzed using output from the present analysis and Ref. 4 to determine if it can withstand penetration loads and if the coring angle is acceptable. Results of these analyses will be discussed in detail in Ref. 1.

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Optimal Correction of Mass and Stiffness Matrices Using Measured Modes

Menahem Baruch*
Technion—Israel Institute of Technology, Haifa, Israel

Nomenclature

A = analytical mass matrix
K = given stiffness matrix
M = optimal mass matrix
N = $A^{1/2}$ or $M^{1/2}$, respectively
p = $Y^{1/2}q$
q = general coordinates vector
R = rigid body modal matrix
T = measured and normalized modal matrix
$\tilde{T}_i$ = ith measured mode shape
T_i = normalized T_i
X = normalized modal matrix
$\tilde{X}$ = modified modal matrix
$\tilde{X}_i$ = ith modified mode shape
Y = corrected stiffness matrix
γ,ϵ = matrix of Lagrange multipliers
Ω^2 = measured frequency matrix
ω_i = measured frequency

Introduction

IN the usual approach to the identification of dynamic structures, vibration tests are used as the only available information. A different approach was proposed in Refs. 1-4 where it was assumed that in addition to vibration tests mass and stiffness matrices obtained by using analytical calculations are also available. A partial performance of this method can be found in Ref. 5. From the vibration tests one usually obtains an incomplete set of natural frequencies and the mode shapes connected with them. Hence, the available data include the analytically obtained mass and stiffness matrices, as well as the measured modes and frequencies. To

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*Professor, Department of Aeronautical Engineering. Member AIAA.

represent a physically possible structure, the data connected with this structure (assumed to be linear) must fulfill the orthogonal condition of the mode shapes. In spite of some expressed doubts about the usefulness of "orthogonality exercises,"[6-10] one cannot overlook the orthogonality requirement. In Refs. 1-4, 11, and 12 the analytically obtained positive definite mass matrix and the measured frequencies are considered to be known quite accurately and therefore to be exact. The mass matrix is used as a reference base. Although the choice of this matrix as the reference base is very convenient, it is not the only possible choice. Any of the given data can be chosen as a reference base. Some discussion on the different possibilities can be found in Refs. 13 and 14. In addition, Berman[13] expresses the opinion that measured data are preferable over some modification of measured data. Following this idea Berman[15] proposes to consider the measured rectangular modal matrix as exact and to modify the analytical mass matrix to comply with the orthogonality requirement. Following Ref. 15, Berman, Wei, and Rao[16] propose a method in which the measured modal matrix is taken as reference base.

Berman[15] modifies the mass matrix in an optimal way using the same weighting matrix used in Refs. 1 and 2 to correct the stiffness matrix. It will be shown here that there exists a different weighting matrix which seems to be more natural. However, the final result is the same. In this Note, as in Refs. 15 and 16, the measured modal matrix is chosen as a reference base. However, the modal matrix is used here as a reference base only after some modification. Thus, two different variations of the method are possible:

1) The first variation[17] is similar to the one given in Refs. 15 and 16 with the exception of semidefinite structures. The mass matrix is supposed to have a higher confidence than the stiffness matrix. First, the analytical mass matrix is corrected to comply with the orthogonality condition in an optimal way by using a natural weighting matrix. The corrected stiffness matrix which satisfies the equilibrium dynamic conditions is again the one given in Refs. 1 and 2.

2) In the second variation of the proposed method the roles of the mass and stiffness matrices are interchanged. Now the stiffness matrix is supposed to be known with a higher confidence than the mass matrix. Hence, the stiffness matrix is corrected in an optimal way to comply with the orthogonality conditions and then the mass matrix is corrected to fulfill the equilibrium dynamic equations.[18]

Structures with Semidefinite Stiffness Matrices

These types of structures are very important in many fields, especially aeronautical and astronautical engineering. Here, in addition to the elastic modes, the structure possesses the so-called rigid body mode shapes. These modes are characterized by zero frequency. The rigid body modes are theoretically well defined without reference to any vibration test. For example, for a two-dimensional representation of an airplane they consist of[19] vertical transition, pitching, and rolling. The parts of the mass matrix connected with the rigid body modes can and must be defined in an independent way. For example, in the case of a two-dimensional representation of an airplane there are the full mass of the airplane, the pitching moment of inertia, and the rolling moment of inertia, respectively. Therefore, we shall keep the parts of the mass matrix connected with the rigid body modes unaffected by the process of correction of the mass matrix. In mathematical form this reads,

$$AR=MR; \qquad R^tAR=R^tMR=I \tag{1}$$

where $R(n\times r)$ represents the theoretically known rigid body modes which have already been orthonormalized. $A(n\times n)$ and $M(n\times n)$ represent the analytical and corrected mass matrix, respectively.

Due to errors in the measurements and to difficulties in the simulation of free-free motion in tests, the measured flexible mode shapes in general will not be orthogonal to the known rigid body modes. Here, we have a clear case where the corrected data are better than the basic data[13,14] and where orthogonality exercises[6,7] are completely necessary. However, following the idea that here the measured mode shapes are the basic reference, one must make only the necessary modifications. Hence, the measured flexible modes will be made orthogonal, in an optimal way, only to the rigid body modes,

$$T_i=\tilde{T}_i(\tilde{T}_i^tA\tilde{T}_i)^{-1/2} \tag{2}$$

where T_i is the normalized mode and $\tilde{T}_i$ the measured mode before normalization.

By minimization of the following norm[1-4]

$$g=\tfrac{1}{2}\|N(\tilde{X}-T)\|; \qquad N=A^{1/2} \tag{3}$$

with the constraint

$$R^tA\tilde{X}=0 \tag{4}$$

one obtains

$$\tilde{X}=[I-RR^tA]T \tag{5}$$

Principally, Eq. (5) represents only a slight necessary modification of the flexible mode shapes. (For more details, see Ref. 17.) In addition, physically possible linear structure must fulfill the following relations

$$MX\Omega^2=YX; \qquad KR=YR=0$$

$$X^tMX=I; \qquad X^tYX=\Omega^2 \tag{6}$$

where $X(n\times m)$ represents a properly normalized modal matrix, $\Omega^2(m\times m)$ is a diagonal matrix that represents the measured frequencies, and $K(n\times n)$ and $Y(n\times n)$ are the analytical and corrected stiffness matrices, respectively.

In the remainder of this Note, the structure will be assumed to have a positive definite stiffness matrix. The validity of the results for semidefinite structures will be explained later.

First Variation

Here

$$X_i=\tilde{X}_i(\tilde{X}_i^tA\tilde{X}_i)^{-1/2} \tag{7}$$

where X_i is the normalized flexible mode and $\tilde{X}_i$ is taken from Eq. (5). We are now looking for a mass matrix $M(n\times n)$ which fulfills the theoretical requirement of orthogonality and is in some sense the closest to the matrix A.

In Refs. 1 and 2 the closeness between two quadratic matrices is measured by a weighted Euclidean norm. Berman[15] takes the weighting matrix to be the analytical mass matrix A. It seems, however, that for the mass matrix connected with a linear dynamic system there exists a natural weighting matrix which arises from the problem itself. The basic vibration equation is given by

$$M\ddot{q}+Yq=0 \tag{8}$$

where $q(n\times 1)$ is the displacement vector and $Y(n\times n)$ the stiffness matrix is assumed to be positive definite. In a manner similar to the one given in Ref. 2 a new vector p is defined by

$$p=Y^{1/2}q \tag{9}$$

Substitution of Eq. (9) into Eq. (8) yields

$$(Y^{-1/2}MY^{-1/2})\ddot{p}+p=0 \tag{10}$$

It is clear that the matrix $(Y^{-1/2}MY^{-1/2})$ governs the vibration equation. It appears that the Euclidean norm of $(Y^{-1/2}MY^{-1/2})$ is the natural weighted norm of the mass matrix.

Minimization of

$$d=\tfrac{1}{2}\|Y^{-1/2}(M-A)Y^{-1/2}\| \tag{11}$$

with the constraint of orthogonality $(X^tMX=I)$ yields

$$M=A-YX(X^tYX)^{-1}(X^tAX-I)(X^tYX)^{-1}X^tY \tag{12}$$

By using the equilibrium equation $(MX\Omega^2=YX)$ one obtains

$$M=A-AX(X^tAX)^{-1}(X^tAX-I)(X^tAX)^{-1}X^tA \tag{13}$$

In spite of using a different weighting matrix, Berman[15] has obtained the same result. During the procedure there appears the inverse of the stiffness matrix. The difficulty for semidefinite structures can be overcome as follows. Assume that the structure is not entirely free but is held by very soft springs. In this case the first frequencies are small but different from zero and the stiffness matrix is nonsingular. After performing the calculations let us assume that the spring constants go to zero and the structure becomes free. One can see that this assumption does not effect the results. To complete the identification of the structure one needs, in addition to Eqs. (1), (2), (5), (7), and (13), the corrected stiffness matrix,[1,2]

$$Y=K-KXX^tM-MXX^tK+MXX^tKXX^tM$$
$$+MX\Omega^2X^tM \tag{14}$$

where $K(n\times n)$ is a given analytical stiffness matrix.

Second Variation

In the first place, the stiffness matrix will be corrected to comply with the orthogonality conditions $(X^tYX=\Omega^2)$. Here, the modal matrix will be normalized with respect to the analytically obtained stiffness matrix

$$X_i=\omega_i\tilde{X}_i(\tilde{X}_i^tK\tilde{X}_i)^{-1/2} \tag{15}$$

where ω_i is the ith measured frequency. Note that normalization [Eq. (15)] is possible only for flexible modes.

Following the explanation given previously, the natural norm to be minimized to obtain an optimal stiffness matrix is given by

$$f=\tfrac{1}{2}\|N^{-1}(Y-K)N^{-1}\|;\qquad N=M^{1/2} \tag{16}$$

Note that the mass matrix M is yet unknown. However, it must satisfy Eq. (6). By using Lagrange multipliers one can incorporate the constraint $(X^tYX=\Omega^2)$ into the norm f. Minimization of the so-obtained Lagrange function in respect to Y yields

$$Y=K-MX(X^tKX-\Omega^2)X^tM \tag{17}$$

By using the equilibrium equation $(MX\Omega^2=YX)$, Eq. (17) can be brought to the form

$$Y=K-YX\Omega^{-2}(X^tKX-\Omega^2)\Omega^{-2}X^tY \tag{18}$$

Multiplication of Eq. (18) by X and a second usage of the equilibrium equation yields finally

$$Y=K-KX(X^tKX)^{-1}(X^tKX-\Omega^2)(X^tKX)^{-1}X^tK \tag{19}$$

Correction of the Mass Matrix

The corrected mass matrix will be found by minimization of the norm

$$d=\tfrac{1}{2}\|Y^{-1/2}(M-A)Y^{-1/2}\| \tag{20}$$

The constraints can be incorporated by using Lagrange multipliers. Minimization with respect to M of the Lagrange function σ obtained in this manner yields

$$\partial\sigma/\partial M=Y^{-1}(M-A)Y^{-1}+2\gamma\Omega^2X^t+2\epsilon=0 \tag{21}$$

where the matrix $2\gamma(n\times m)$ is connected with the equilibrium constraint $(MX\Omega^2=YX)$ and $\epsilon(n\times n)$ is connected with the constraint of the symmetry of the mass matrix $(M^t=M)$. By using the antisymmetric properties of ϵ one can remove it from Eq. (21) to obtain

$$M=A-Y\gamma\Omega^2X^tY-YX\Omega^2\gamma^tY \tag{22}$$

Now, a crucial assumption connected with the matrix γ will be made

$$\Omega^4\gamma^tYX=X^tY\gamma\Omega^4 \tag{23}$$

Note that Eq. (23) and the antisymmetric properties of ϵ reduce the independent variables of the matrix M to the number of independent members reduced by the constraints. In this way the problem becomes well defined. In addition one can easily verify the following equality

$$(I+X\Omega^{-2}X^tY)^{-1}=I-\tfrac{1}{2}X\Omega^{-2}X^tY \tag{24}$$

From Eqs. (6) and (22-24) one obtains

$$\gamma=Y^{-1}AX\Omega^{-4}-\tfrac{1}{2}X\Omega^{-6}-\tfrac{1}{2}X\Omega^{-2}X^tAX\Omega^{-4} \tag{25}$$

It can be shown that Eq. (25) satisfies Eq. (23).

Substitution of Eq. (25) into Eq. (22) yields finally[18]

$$M=A-AX\Omega^{-2}X^tY-YX\Omega^{-2}X^tA+YX\Omega^{-4}X^tY$$
$$+YX\Omega^{-2}X^tAX\Omega^{-2}X^tY \tag{26}$$

Equation (26) completes the second variation of the method of reference base in which the base is the measured modal matrix.

Conclusions

Two different variations of a method for identification of linear dynamic structures were described. In both variations the reference base is the measured modal matrix. It was shown that in the identification process for structures with semidefinite matrix the measured mode shapes cannot be kept unchanged. They have to be slightly modified. Only then can they be used to correct the given analytical mass and stiffness matrices. In the first variation the mass matrix, which is considered to be known with higher confidence, is corrected to meet the orthogonality conditions and then the stiffness matrix is corrected to satisfy the dynamic equations. In the second variation the stiffness matrix, which is now considered to be known with higher confidence, is corrected to meet the orthogonality conditions and then the mass matrix is corrected to satisfy the dynamic equations.

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Galerkin Method and Axisymmetric Vibrations of Polar-Orthotropic Circular Plates

D. Avalos* and H. Hack*
Universidad Nacional de Mar del Plata
Mar del Plata, Argentina
and
P. A. A. Laura†
Institute of Applied Mechanics
Base Naval Puerto Belgrano, Argentina

Nomenclature

A_i, A_j	= undetermined constants
a	= radius of plate
$D_r, D_\theta, D_{r\theta}$	$= E_r(h^3/12), E_\theta(h^3/12), E_{r\theta}(h^3/12)$
$E_r, E_\theta, E_{r\theta}$	= elastic constants of the plate material
h	= thickness
k	$= D_\theta / D_r$
r, θ	= polar coordinates
W	= transverse deflection amplitude
W_a	= approximate transverse deflection amplitude
ρ	= mass density of the plate material
ω	= circular frequency
Ω_{oo}	= fundamental frequency coefficient $[=\sqrt{(\rho h/D_r)}\,\omega_{oo}a^2]$

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*Professor, School of Engineering.
†Director and Research Scientist.

Introduction

TRANSVERSE, axisymmetric vibrations of circular plates of polar orthotropy have been analyzed by several authors.[1-3] However, several inconsistencies appear in these studies, as has been first pointed out by Leissa[4] and discussed in greater detail by Prathap and Varadan.[5] These authors point out that the inconsistencies are generated because certain boundary conditions are ignored or treated incorrectly in Refs. 1-3 when using the Galerkin method. Prathap and Varadan[5] show that, in the case of a clamped orthotropic plate, convergence is achieved if the displacement amplitude is approximated using a polynomial approach, i.e.,

$$W_a = \sum_{i=0}^{I} A_i \left[1 - \left(\frac{r}{a}\right)^2\right]^{2+i} \tag{1}$$

and generating the frequency equation by means of the Lagrangian formulation.

Conversely, it is shown in Ref. 6 that polynomial coordinate functions yield excellent accuracy in the case of elastically restrained plates of circular orthotropy subject to an in-plane state of hydrostatic stress when the Ritz formulation is used.

It is the object of this study to show the existence of a type of coordinate, polynomial functions that, in spite of the fact that they do not comply with the "internal" condition[5]:

$$(1-k^2)\frac{d^2 W}{dr^2}(O) = 0, \qquad k = \frac{D_\theta}{D_r} \tag{2}$$

converge in a very satisfactory fashion when the Galerkin formulation is invoked.

Discussion of the Methodology and Numerical Results

If one considers the response of the orthotropic, circular plate subjected to a uniformly distributed transverse load, one has the following functional relations[7] for the displacement function $w(r)$:

$$w(r) = A + Cr^{1+k} + qr^4/8(9-k^2)D_r \tag{3}$$

where A and C are determined using the governing boundary conditions.

One notices immediately the existence of the term r^{1+k}. Accordingly, it seems reasonable to try coordinate functions where the parameter k is contained in a similar fashion.

It was found convenient to use the approximation:

$$W(r) \simeq W_a(r) = \sum_{j=0}^{J} A_j \left[\alpha_j\left(\frac{r}{a}\right)^{3+k} + \beta_j\left(\frac{r}{a}\right)^{1+k} + 1\right]\left(\frac{r}{a}\right)^{4j} \tag{4}$$